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Finite and Infinite Integral Formulas Associated with a Family of Incomplete *I*- Functions

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***Abstract:** In recent years, research focuses on the integral representations of several kinds of special functions. In this paper, first we establish the integral representation of incomplete I- functions. Further, we find out some special cases of these integrals. Finally, we derived certain integrals involving a product of incomplete I- function and some other special functions.2010 Mathematics Subject Classification: 33E20, 44A40.*

***Keywords:** and phrases: Incomplete I-functions; Incomplete Gamma functions and Mellin- Barnes type integral.*

I. INTRODUCTION

In the last decade, many authors (see, e.g. [1-7], [11-14]) have developed numerous integral formulas involving a variety of incomplete hypergeometric functions. Such integral formulas have many applications in potential field of physics, applied sciences, engineering and chemical sciences.

Recently, Bansal et al. [1] introduce new incomplete *I*-functions which is an extension of the Saxena's *I*- function [10] and gave certain interesting integral formulas and transform of these functions, which are expressed in terms of generalized (Wright) hypergeometric function.

The incomplete *I*- functions (IIFs) ${}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}(w)$ and ${}^{(\gamma)}I^{m,n}_{p_l,q_l;r}(w)$ [1] are defined as follows

$$ {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}(w) = {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}\left[w\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right] $$

$$ = \frac{1}{2\pi\omega}\int_L \emptyset_1(\xi,y)\ \ w^{-\xi}d\xi, $$

where

$$ \emptyset_1(\xi,y) = \frac{\Gamma(1-g_1-\boldsymbol{G}_1\xi,y)\prod_{j=1}^{m}\Gamma\left(h_j+\boldsymbol{H}_j\xi\right)\prod_{j=2}^{n}\Gamma\left(1-g_j-\boldsymbol{G}_j\xi\right)}{\sum_{l=1}^{r}\left[\prod_{j=m+1}^{q_l}\Gamma\left(1-h_{jl}-\boldsymbol{H}_{jl}\xi\right)\prod_{j=n+1}^{p_l}\Gamma\left(g_{jl}+\boldsymbol{G}_{jl}\xi\right)\right]} \quad (1.2) $$

and

$$ {}^{(\gamma)}I^{m,n}_{p_l,q_l;r}(w) = {}^{(\gamma)}I^{m,n}_{p_l,q_l;r}\left[w\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right] $$

$$ = \frac{1}{2\pi\omega}\int_L \emptyset_2(\xi,y)\ \ w^{-\xi}d\xi \quad (1.3) $$

where

$$ \emptyset_2(\xi,y) = \frac{\gamma(1-g_1-\boldsymbol{G}_1\xi,y)\prod_{j=1}^{m}\Gamma\left(h_j+\boldsymbol{H}_j\xi\right)\prod_{j=2}^{n}\Gamma\left(1-g_j-\boldsymbol{G}_j\xi\right)}{\sum_{l=1}^{r}\left[\prod_{j=m+1}^{q_l}\Gamma\left(1-h_{jl}-\boldsymbol{H}_{jl}\xi\right)\prod_{j=n+1}^{p_l}\Gamma\left(g_{jl}+\boldsymbol{G}_{jl}\xi\right)\right]} \quad (1.4) $$

The incomplete *I*- functions ${}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}(w)$ and ${}^{(\gamma)}I^{m,n}_{p_l,q_l;r}(w)$ in (1.1) and (1.3) exists for $y \geq 0$ under the following set of conditions satisfied.

The contour L in the complex ξ-plane extends from $c - i\infty$ to $c + i\infty$, $c \in Re$ and poles of the gamma functions $\Gamma\left(1-g_j-\boldsymbol{G}_j\xi\right), j=\overline{1,n}$ do not exactly match with the poles of the gamma functions $\Gamma\left(h_j+\boldsymbol{H}_j\xi\right), j=\overline{1,m}$. The parameters m,n,p_l,q_l are non negative integers satisfying $0 \leq n \leq p_l, 0 \leq m \leq q_l, l=\overline{1,r}$. The parameters $\boldsymbol{G}_j,\boldsymbol{H}_j,\boldsymbol{G}_{jl},\boldsymbol{H}_{jl}$ are positive integers and g_j,h_j,g_{jl},h_{jl} are complex. All poles of $\emptyset_1(\xi,y)$and $\emptyset_2(\xi,y)$ are supposed to be simple and the empty product is treated as unity.

(i) $\quad \lambda_l > 0,\ \ |\arg(w)| < \frac{\pi}{2}\lambda_l$,

(ii) $\quad \lambda_l \geq 0,\ \ |\arg(w)| \leq \frac{\pi}{2}\lambda_l$ and $\mathrm{Re}\ (\mu_l+1) < 0$,

where

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$$\lambda_l = \sum_{j=2}^{n} \boldsymbol{G}_j + \sum_{j=2}^{n} \boldsymbol{H}_j - \sum_{j=n+1}^{p_l} \boldsymbol{G}_{jl} - \sum_{j=m+1}^{q_l} \boldsymbol{H}_{jl} \quad (1.5)$$

and

$$\mu_l = \sum_{j=2}^{n} g_j - \sum_{j=2}^{n} h_j + \sum_{j=n+1}^{p_l} g_{jl} - \sum_{j=m+1}^{q_l} h_{jl} + \frac{1}{2}(p_l - q_l), \quad (1.6)$$

for all $l = \overline{1,r}$.

Now, we call here Mellintransformof incomplete *I*-functions, which were given by Bansal et al. (see [1], p.1251, equation (3.2)).

Mellin Transform

If

$\lambda_l > 0, \tau > 0, |\arg(w)| < \frac{\pi}{2}\lambda_l$, Re $(\mu_l) + 1 < 0$, $l = \overline{1,r}$

$$y \geq 0, \quad k > 0, \quad -\min_{1\leq j\leq m} \mathrm{Re}\left(\frac{h_j}{\boldsymbol{H}_j}\right) < \mathrm{Re}(þ) < \min_{1\leq j\leq n} \mathrm{Re}\left(\frac{1-g_j}{\boldsymbol{G}_j}\right),$$

Then Mellin transform of incomplete *I*- functions ${}^{(\Gamma)}I_{p_l,q_l;r}^{m,n}(w)$ and ${}^{(\gamma)}I_{p_l,q_l;r}^{m,n}(w)$ defined as follows:

$$\mathfrak{M}\left\{ {}^{(\Gamma)}I_{p_l,q_l;r}^{m,n}\left[kw^{\tau} \left| \begin{matrix} (g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right]; þ \right\} = \frac{k^{-þ}}{\tau} \emptyset_1\left(\frac{þ}{\tau}, y\right) \quad (1.7)$$

and

$$\mathfrak{M}\left\{ {}^{(\gamma)}I_{p_l,q_l;r}^{m,n}\left[kw \left| \begin{matrix} (g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right]; þ \right\} = k^{-þ}\emptyset_2(þ, y) \quad (1.8)$$

where$\emptyset_1(þ, y)$and $\emptyset_2(þ, y)$ are given in (1.2) and (1.4) respectively and provided that each member in (1.7) and (1.8) exist.

If we take $y = 0$ in (1.7),then Mellin transform of *I*-function [10] is defined as follows:

$$\mathfrak{M}\left\{ I_{p_l,q_l;r}^{m,n}\left[kw^{\tau} \left| \begin{matrix} (g_j,\boldsymbol{G}_j)_{1,n}(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right]; þ \right\} = \frac{k^{-þ}}{\tau}\phi\left(\frac{þ}{\tau}\right) \quad (1.9)$$

provided that each member in (1.9) exist.

II. INTEGRAL REPRESENTATIONS OF INCOMPLETE I-FUNCTIONS

In recent year, Srivastava et al. [13] and Bansal et al. [2] established the integral representation of the incomplete Gauss hypergeometric functions and incomplete H-functions respectively. So motivated by their work, we establish the integral representation of incomplete *I*- functions defined in (1.1) and (1.3).

Theorem I

If $y \geq 0$ *and* $Re(g_1) > 0$, *thenintegral representation ofincomplete I- functions*${}^{(\Gamma)}I_{p_l,q_l;r}^{m,n}(w)$ *and* ${}^{(\gamma)}I_{p_l,q_l;r}^{m,n}(w)$ *(1.1) and (1.3) in following way:*

$$ {}^{(\Gamma)}I_{p_l,q_l;r}^{m,n}\left[w \left| \begin{matrix} (1-g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right] = \int_y^{\infty} t^{g_1-1}e^{-t} I_{p_l-1,q_l;r}^{m,n-1}\left[t^{\boldsymbol{G}_1}w \left| \begin{matrix} (g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right] dt \quad (2.1)$$

and

$$ {}^{(\gamma)}I_{p_l,q_l;r}^{m,n}\left[w \left| \begin{matrix} (1-g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right] = \int_0^{y} t^{g_1-1}e^{-t} I_{p_l-1,q_l;r}^{m,n-1}\left[t^{\boldsymbol{G}_1}w \left| \begin{matrix} (g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l} \\ (h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l} \end{matrix} \right. \right] dt \quad (2.2)$$

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Proof:

To prove the assertion (2.1), firstly we express the Mellin-Barnes contour integral form of well knownSaxena's*I*-function [10] in R.H.S of (2.1), we get

$$\int_y^{\infty} t^{g_1-1}e^{-t}\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma(h_j+\boldsymbol{H}_j\xi)\prod_{j=2}^{n}\Gamma(1-g_j-\boldsymbol{G}_j\xi)}{\sum_{l=1}^{r}\left[\prod_{j=m+1}^{q_l}\Gamma(1-h_{jl}-\boldsymbol{H}_{jl}\xi)\prod_{j=n+1}^{p_l}\Gamma(g_{jl}+\boldsymbol{G}_{jl}\xi)\right]}(t^{\boldsymbol{G}_1}w)^{-\xi}d\xi\ dt\ , \tag{2.3}$$

Further, changing the order of integration and with the help of familiar upper incomplete Gamma function definition (see [1], equation (1.4)).We obtain

$$\frac{1}{2\pi\omega}\int_L \frac{\Gamma(g_1-\boldsymbol{G}_1\xi,y)\prod_{j=1}^{m}\Gamma(h_j+\boldsymbol{H}_j\xi)\prod_{j=2}^{n}\Gamma(1-g_j-\boldsymbol{G}_j\xi)}{\sum_{l=1}^{r}\left[\prod_{j=m+1}^{q_l}\Gamma(1-h_{jl}-\boldsymbol{H}_{jl}\xi)\prod_{j=n+1}^{p_l}\Gamma(g_{jl}+\boldsymbol{G}_{jl}\xi)\right]}w^{-\xi}d\xi$$

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with the help of (1.1),we arrive at the result of (2.1).
Similarly, we get the integral representation (2.2) of incomplete *I*- function$^{(\gamma)}I_{p_l,q_l;r}^{m,n}(w)$with the help of lower incomplete Gamma function definition (see [1], equation (1.3)).

2.1 Special cases:

(i) If we set $y=0$ in (2.1), then we obtain integral representation of Saxena's*I*- function [10] as

$$I_{p_l,q_l;r}^{m,n}\left[w\left|\begin{matrix}(1-g_1,\boldsymbol{G}_1),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\(h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right]$$
$$=\int_0^{\infty}t^{g_1-1}e^{-t}I_{p_l-1,q_l;r}^{m,n-1}\left[t^{\boldsymbol{G}_1}w\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\(h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dt \tag{2.1.1}$$

(i) Again, setting $r=1$ in (2.1) and (2.2), then we obtain integral representation of incomplete *H*- function[14], which is established by Bansal et al. [2]

$$\Gamma_{p,q}^{m,n}\left[w\left|\begin{matrix}(1-g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]$$
$$=\int_y^{\infty}t^{g_1-1}e^{-t}H_{p-1,q}^{m,n-1}\left[t^{\boldsymbol{G}_1}w\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]dt \tag{2.1.2}$$

and

$$\gamma_{p,q}^{m,n}\left[w\left|\begin{matrix}(1-g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]$$
$$=\int_0^{y}t^{g_1-1}e^{-t}H_{p-1,q}^{m,n-1}\left[t^{\boldsymbol{G}_1}w\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]dt \tag{2.1.3}$$

(ii) If we take $r=1$ and $y=0$, then (2.1) reduces to familiar Fox's *H*-function ([8], [9], [15]):

$$H_{p,q}^{m,n}\left[w\left|\begin{matrix}(1-g_1,\boldsymbol{G}_1),(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]$$
$$=\int_0^{\infty}t^{g_1-1}e^{-t}H_{p-1,q}^{m,n-1}\left[t^{\boldsymbol{G}_1}w\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{2,p}\\(h_j,\boldsymbol{H}_j)_{1,q}\end{matrix}\right.\right]dt \tag{2.1.4}$$

III. INTEGRAL FORMULAS INVOLVING A PRODUCT OF INCOMPLETE *I*-FUNCTION AND THE VARIOUS KINDS OF SPECIAL FUNCTIONS

Theorem II

If

$$\sigma>0,\qquad -\sigma\max_{1\le j\le n}Re\left(\frac{1-g_j}{\boldsymbol{G}_j}\right)-\min_{1\le j\le M}Re\left(\frac{b_j}{\beta_j}\right)<Re(\rho)<\sigma\min_{1\le j\le m}Re\left(\frac{h_j}{\boldsymbol{H}_j}\right)+\max_{1\le j\le N}Re\left(\frac{1-a_j}{\alpha_j}\right),$$

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Then the following improper integral holds for $y \geq 0$:

$$\int_0^\infty w^{\rho-1}\, {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}\left[kw^{-\sigma}\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\(h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right] \times I^{M,N}_{P_l,Q_l;R}\left[sw\left|\begin{matrix}(a_j,\alpha_j)_{1,N};\ (a_{ji},\alpha_{ji})_{N+1,P_l}\\(b_j,\beta_j)_{1,M};\ (b_{ji},\beta_{ji})_{M+1,Q_l}\end{matrix}\right.\right]dw = s^{-\rho}\, {}^{(\Gamma)}I^{m+M,n+N}_{p_l+P_l,q_l+Q_l;rR}\left[ks^{\sigma}\left|\begin{matrix}C^{\cdot}\\D^{\cdot}\end{matrix}\right.\right] \quad (3.1)$$

and

$$\int_0^\infty w^{\rho-1}\, {}^{(\gamma)}I^{m,n}_{p_l,q_l;r}\left[kw^{-\sigma}\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,n},(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l}\\(h_j,\boldsymbol{H}_j)_{1,m},(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l}\end{matrix}\right.\right] \times I^{M,N}_{P_l,Q_l;R}\left[sw\left|\begin{matrix}(a_j,\alpha_j)_{1,N};\ (a_{ji},\alpha_{ji})_{N+1,P_l}\\(b_j,\beta_j)_{1,M};\ (b_{ji},\beta_{ji})_{M+1,Q_l}\end{matrix}\right.\right]dw$$

$$= s^{-\rho}\, {}^{(\gamma)}I^{m+M,n+N}_{p_l+P_l,q_l+Q_l;rR}\left[ks^{\sigma}\left|\begin{matrix}C^{\cdot}\\D^{\cdot}\end{matrix}\right.\right] (3.2)$$

where

$C^{\cdot} = (g_1,\boldsymbol{G}_1,y),(a_j+\rho\alpha_j,\sigma\alpha_j)_{1,N},(g_j,\boldsymbol{G}_j)_{2,n};(g_{jl},\boldsymbol{G}_{jl})_{n+1,p_l},(a_{ji}+\rho\alpha_{ji},\sigma\alpha_{ji})_{N+1,P_l}$

and

$D^{\cdot} = (b_j+\beta_j\rho,\sigma\beta_j)_{1,M},(h_j,\boldsymbol{H}_j)_{1,m};(h_{jl},\boldsymbol{H}_{jl})_{m+1,q_l},(b_{ji}+\rho\beta_{ji},\sigma\beta_{ji})_{M+1,Q_l}$

provided that the conditions of incomplete I-function and I-function are satisfied.

Proof:

To prove the assertion (3.1), first we write Mellin contour integral form of incomplete *I*-function with the help of (1.1), we obtain (sayΞ)

$$\Xi = \int_0^\infty w^{\rho-1}\left[\frac{1}{2\pi\omega}\int_L \emptyset_2(\xi,y)\ (kw^{-\sigma})^{-\xi}d\xi\right] I^{M,N}_{P_l,Q_l;R}\left[sw\left|\begin{matrix}(a_j,\alpha_j)_{1,N};\ (a_{ji},\alpha_{ji})_{N+1,P_l}\\(b_j,\beta_j)_{1,M};\ (b_{ji},\beta_{ji})_{M+1,Q_l}\end{matrix}\right.\right]dw \quad (3.3)$$

Now, changing the order of integration and using Mellin transform of *I*-function, we get

$$\Xi = \frac{s^{-\rho}}{2\pi\omega}\int_L \emptyset_1(\xi,y)\phi(\rho+\sigma\xi)\ (ks^{\sigma})^{-\xi}d\xi \quad (3.4)$$

Finally, with the help of (1.1) and (see [10], equation (2.1.42)) we get the desired result.

Theorem III

If

$$\sigma > 0, \quad -\sigma \max_{1\leq j\leq N} Re\left(\frac{1-g_j}{\boldsymbol{G}_j}\right) - \min_{1\leq j\leq m} Re\left(\frac{f_j}{\boldsymbol{F}_j}\right) < Re(\rho) < \sigma \min_{1\leq j\leq M} Re\left(\frac{h_j}{\boldsymbol{H}_j}\right) + \max_{1\leq j\leq n} Re\left(\frac{1-e_j}{\boldsymbol{E}_j}\right),$$

Then the following improper integral holds for $x, y \geq 0$:

$$\int_0^\infty w^{\rho-1}\, {}^{(\Gamma)}I^{M,N}_{P_l,Q_l;R}\left[kw^{-\sigma}\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,N},(g_{jl},\boldsymbol{G}_{jl})_{N+1,P_l}\\(h_j,\boldsymbol{H}_j)_{1,M},(h_{jl},\boldsymbol{H}_{jl})_{M+1,Q_l}\end{matrix}\right.\right] {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r}\left[sw^{\tau}\left|\begin{matrix}(e_1,\boldsymbol{E}_1,x),(e_j,\boldsymbol{E}_j)_{2,n},(e_{jl},\boldsymbol{E}_{jl})_{n+1,p_l}\\(f_j,\boldsymbol{F}_j)_{1,m},(f_{jl},\boldsymbol{F}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dw$$

$$= \frac{s^{-\rho}}{\tau}\, {}^{(\Gamma)}I^{M+m,N+n}_{P_l+p_l,Q_l+q_l;Rr}\left[ks^{\sigma}\left|\begin{matrix}C^{\cdot}\\D^{\cdot}\end{matrix}\right.\right] \quad (3.5)$$

and

$$\int_0^\infty w^{\rho-1}\, {}^{(\gamma)}I^{M,N}_{P_l,Q_l;R}\left[kw^{-\sigma}\left|\begin{matrix}(g_1,\boldsymbol{G}_1,y),(g_j,\boldsymbol{G}_j)_{2,N},(g_{jl},\boldsymbol{G}_{jl})_{N+1,P_l}\\(h_j,\boldsymbol{H}_j)_{1,M},(h_{jl},\boldsymbol{H}_{jl})_{M+1,Q_l}\end{matrix}\right.\right] {}^{(\gamma)}I^{m,n}_{p_l,q_l;r}\left[sw^{\tau}\left|\begin{matrix}(e_1,\boldsymbol{E}_1,x),(e_j,\boldsymbol{E}_j)_{2,n},(e_{jl},\boldsymbol{E}_{jl})_{n+1,p_l}\\(f_j,\boldsymbol{F}_j)_{1,m},(f_{jl},\boldsymbol{F}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dw$$

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$$= \frac{s^{-\rho}}{\tau} {}^{(\gamma)}I^{M+m,N+n}_{P_l+p_l,Q_l+q_l;Rr} \left[ks^{\sigma} \middle| \begin{matrix} C^{\cdot} \\ D^{\cdot} \end{matrix} \right] \tag{3.6}$$

where

$$C^{\cdot} = \left(e_1 + \frac{\rho}{\tau}\boldsymbol{E}_1, \frac{\sigma}{\tau}\boldsymbol{E}_1, x\right)(g_1, \boldsymbol{G}_1, y), \left(e_j + \frac{\rho}{\tau}\boldsymbol{E}_j, \frac{\sigma}{\tau}\boldsymbol{E}_j\right)_{2,n}, (g_j, \boldsymbol{G}_j)_{2,N}; (g_{jl}, \boldsymbol{G}_{jl})_{N+1,P_l}, \left(e_{jl} + \frac{\rho}{\tau}\boldsymbol{E}_{jl}, \frac{\sigma}{\tau}\boldsymbol{E}_{jl}\right)_{n+1,p_l}$$

and

$$D^{\cdot} = \left(f_j + \frac{\rho}{\tau}\boldsymbol{F}_j, \frac{\sigma}{\tau}\boldsymbol{F}_j\right)_{1,m}, (h_j, \boldsymbol{H}_j)_{1,M}; (h_{jl}, \boldsymbol{H}_{jl})_{M+1,Q_l}, \left(f_{jl} + \frac{\rho}{\tau}\boldsymbol{F}_{jl}, \frac{\sigma}{\tau}\boldsymbol{F}_{jl}\right)_{m+1,q_l}$$

provided that the conditions of incomplete I-functions in (1.1) and (1.3) are satisfied.

Proof:

To prove the assertion (3.5), first we write Mellin contour integral form of incomplete *I*-function with the help of (1.1), we obtain (sayΩ)

$$\Omega = \int_0^{\infty} w^{\rho-1} \left[\frac{1}{2\pi\omega} \int_L \emptyset_1(\xi, y) \ (kw^{-\sigma})^{-\xi} d\xi \right]$$

$$ {}^{(\Gamma)}I^{M,N}_{P_l,Q_l;R} \left[sw^{\tau} \middle| \begin{matrix} (e_1, \boldsymbol{E}_1, x), (e_j, \boldsymbol{E}_j)_{2,n}, (e_{jl}, \boldsymbol{E}_{jl})_{n+1,p_l} \\ (f_j, \boldsymbol{F}_j)_{1,m}, (f_{jl}, \boldsymbol{F}_{jl})_{m+1,q_l} \end{matrix} \right] dw \tag{3.7}$$

Now, changing the order of integration and using Mellin transform of incomplete *I*-function (1.7), we get

$$\Omega = \frac{s^{-\rho}}{\tau(2\pi\omega)} \int_L \emptyset_1(\xi, y) \phi_2\left(\frac{\rho + \sigma\xi}{\tau}, x\right) \ (ks^{\sigma})^{-\xi} d\xi \tag{3.8}$$

Finally, with the help of (1.1), we get the desired result (3.5) of Theorem III.

Theorem IV

If

$$\sigma > 0, \quad -\sigma \max_{1 \le j \le N} Re\left(\frac{1 - g_j}{\boldsymbol{G}_j}\right) - \min_{1 \le j \le m} Re\left(\frac{f_j}{\boldsymbol{F}_j}\right) < Re(\rho) < \sigma \min_{1 \le j \le M} Re\left(\frac{h_j}{\boldsymbol{H}_j}\right) + \max_{1 \le j \le n} Re\left(\frac{1 - e_j}{\boldsymbol{E}_j}\right),$$

Then the following improper integral holds for $x, y \ge 0$*:*

$$\int_0^{\infty} w^{\rho-1} \Gamma^{M,N}_{P,Q} \left[kw^{-\sigma} \middle| \begin{matrix} (g_1, \boldsymbol{G}_1, y), (g_j, \boldsymbol{G}_j)_{2,P} \\ (h_j, \boldsymbol{H}_j)_{1,Q} \end{matrix} \right]$$

$$ {}^{(\Gamma)}I^{m,n}_{p_l,q_l;r} \left[sw^{\tau} \middle| \begin{matrix} (e_1, \boldsymbol{E}_1, x), (e_j, \boldsymbol{E}_j)_{2,n}, (e_{jl}, \boldsymbol{E}_{jl})_{n+1,p_l} \\ (f_j, \boldsymbol{F}_j)_{1,m}, (f_{jl}, \boldsymbol{F}_{jl})_{m+1,q_l} \end{matrix} \right] dw$$

$$= \frac{s^{-\rho}}{\tau} {}^{(\Gamma)}I^{M+m,N+n}_{P+p_l,Q+q_l;r} \left[ks^{\sigma} \middle| \begin{matrix} C^{\cdot} \\ D^{\cdot} \end{matrix} \right] \tag{3.9}$$

and

$$\int_0^{\infty} w^{\rho-1} \gamma^{M,N}_{P,Q} \left[kw^{-\sigma} \middle| \begin{matrix} (g_1, \boldsymbol{G}_1, y), (g_j, \boldsymbol{G}_j)_{2,P} \\ (h_j, \boldsymbol{H}_j)_{1,Q} \end{matrix} \right]$$

$$ {}^{(\gamma)}I^{m,n}_{p_l,q_l;r} \left[sw^{\tau} \middle| \begin{matrix} (e_1, \boldsymbol{E}_1, x), (e_j, \boldsymbol{E}_j)_{2,n}, (e_{jl}, \boldsymbol{E}_{jl})_{n+1,p_l} \\ (f_j, \boldsymbol{F}_j)_{1,m}, (f_{jl}, \boldsymbol{F}_{jl})_{m+1,q_l} \end{matrix} \right] dw$$

$$= \frac{s^{-\rho}}{\tau} {}^{(\gamma)}I^{M+m,N+n}_{P+p_l,Q+q_l;r} \left[ks^{\sigma} \middle| \begin{matrix} C^{\cdot} \\ D^{\cdot} \end{matrix} \right] \tag{3.10}$$

where

$$C^{\cdot} = \left(e_1 + \frac{\rho}{\tau}\boldsymbol{E}_1, \frac{\sigma}{\tau}\boldsymbol{E}_1, x\right)(g_1, \boldsymbol{G}_1, y), \left(e_j + \frac{\rho}{\tau}\boldsymbol{E}_j, \frac{\sigma}{\tau}\boldsymbol{E}_j\right)_{2,n}, (g_j, \boldsymbol{G}_j)_{2,P}, \left(e_{jl} + \frac{\rho}{\tau}\boldsymbol{E}_{jl}, \frac{\sigma}{\tau}\boldsymbol{E}_{jl}\right)_{n+1,p_l}$$

and

$$D^{\cdot} = \left(f_j + \frac{\rho}{\tau}\boldsymbol{F}_j, \frac{\sigma}{\tau}\boldsymbol{F}_j\right)_{1,m}, (h_j, \boldsymbol{H}_j)_{1,Q}, \left(f_{jl} + \frac{\rho}{\tau}\boldsymbol{F}_{jl}, \frac{\sigma}{\tau}\boldsymbol{F}_{jl}\right)_{m+1,q_l}$$

provided that the conditions of incomplete I-function and incomplete H-function [14] are satisfied respectively.

Proof:

To prove the assertion (3.9), first we write Mellin contour integral form of incomplete *H*-function with the help of (see [2], equation (4)), we obtain (sayΘ)

$$\Theta = \int_0^\infty w^{\rho-1}\left[\frac{1}{2\pi\omega}\int_L \ \Xi_1(\xi,y) \ \ (kw^{-\sigma})^{-\xi} d\xi\right]$$

$$^{(\Gamma)}I^{M,N}_{P_l,Q_l;R}\left[sw^\tau\left|\begin{matrix}(e_1,\boldsymbol{E}_1,x),(e_j,\boldsymbol{E}_j)_{2,n},(e_{jl},\boldsymbol{E}_{jl})_{n+1,p_l}\\ (f_j,\boldsymbol{F}_j)_{1,m},(f_{jl},\boldsymbol{F}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dw, \qquad (3.11)$$

Now, changing the order of integration and using Mellin transform of incomplete *I*-function (1.7), we get

$$\Theta = \frac{s^{-\rho}}{\tau(2\pi\omega)}\int_L \ \Xi_1(\xi,y)\phi_1\left(\frac{\rho+\sigma\xi}{\tau},x\right) \ \ (ks^\sigma)^{-\xi} d\xi (3.12)$$

Finally, with the help of (1.1), we get the desired result.

Theorem V

If

$$\sigma > 0, \quad -\sigma \max_{1\le j\le N} Re\left(\frac{1-g_j}{\boldsymbol{G}_j}\right) - \min_{1\le j\le m} Re\left(\frac{f_j}{\boldsymbol{F}_j}\right) < Re(\rho) < \sigma \min_{1\le j\le M} Re\left(\frac{h_j}{\boldsymbol{H}_j}\right) + \max_{1\le j\le n} Re\left(\frac{1-e_j}{\boldsymbol{E}_j}\right),$$

Then the following improper integral holds for $x \geq 0$*:*

$$\int_0^\infty w^{\rho-1}H^{M,N}_{P,Q}\left[kw^{-\sigma}\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{1,P}\\ (h_j,\boldsymbol{H}_j)_{1,Q}\end{matrix}\right.\right]$$

$$^{(\Gamma)}I^{m,n}_{p_l,q_l;r}\left[sw^\tau\left|\begin{matrix}(e_1,\boldsymbol{E}_1,x),(e_j,\boldsymbol{E}_j)_{2,n},(e_{jl},\boldsymbol{E}_{jl})_{n+1,p_l}\\ (f_j,\boldsymbol{F}_j)_{1,m},(f_{jl},\boldsymbol{F}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dw$$

$$= \frac{s^{-\rho}}{\tau} {}^{(\Gamma)}I^{M+m,N+n}_{P+p_l,Q+q_l;r}\left[ks^\sigma\left|\begin{matrix}C^{\cdot}\\ D^{\cdot}\end{matrix}\right.\right] \qquad (3.13)$$

and

$$\int_0^\infty w^{\rho-1}H^{M,N}_{P,Q}\left[kw^{-\sigma}\left|\begin{matrix}(g_j,\boldsymbol{G}_j)_{1,P}\\ (h_j,\boldsymbol{H}_j)_{1,Q}\end{matrix}\right.\right]$$

$$^{(\gamma)}I^{m,n}_{p_l,q_l;r}\left[sw^\tau\left|\begin{matrix}(e_1,\boldsymbol{E}_1,x),(e_j,\boldsymbol{E}_j)_{2,n},(e_{jl},\boldsymbol{E}_{jl})_{n+1,p_l}\\ (f_j,\boldsymbol{F}_j)_{1,m},(f_{jl},\boldsymbol{F}_{jl})_{m+1,q_l}\end{matrix}\right.\right]dw$$

$$= \frac{s^{-\rho}}{\tau} {}^{(\gamma)}I^{M+m,N+n}_{P+p_l,Q+q_l;r}\left[ks^\sigma\left|\begin{matrix}C^{\cdot}\\ D^{\cdot}\end{matrix}\right.\right] \qquad (3.14)$$

where

$$C^{\cdot} = \left(e_1+\frac{\rho}{\tau}\boldsymbol{E}_1,\frac{\sigma}{\tau}\boldsymbol{E}_1,x\right),\left(e_j+\frac{\rho}{\tau}\boldsymbol{E}_j,\frac{\sigma}{\tau}\boldsymbol{E}_j\right)_{2,n},(g_j,\boldsymbol{G}_j)_{1,P},\left(e_{jl}+\frac{\rho}{\tau}\boldsymbol{E}_{jl},\frac{\sigma}{\tau}\boldsymbol{E}_{jl}\right)_{n+1,p_l}$$

and

$$D^{\cdot} = \left(f_j+\frac{\rho}{\tau}\boldsymbol{F}_j,\frac{\sigma}{\tau}\boldsymbol{F}_j\right)_{1,m},(h_j,\boldsymbol{H}_j)_{1,Q},\left(f_{jl}+\frac{\rho}{\tau}\boldsymbol{F}_{jl},\frac{\sigma}{\tau}\boldsymbol{F}_{jl}\right)_{m+1,q_l}$$

provided that the conditions of incomplete I-function [1] and Fox'sH-function(see [8], [9]) are satisfied.

Proof: In the similar manner of the Theorem IV, and using the result of equation (3.9) and (3.10), we obtain the result (3.13) and (3.14) of the Theorem V.

REFERENCES

1. M. K. Bansal and D.Kumar (2020). On the integral operators pertaining to a family of incomplete *I*-function, *AIMS Mathematics*, **5**(2), 1247-1259. [CrossRef]
2. Bansal, M. K., Kumar D., Singh, J. and Nisar, K. S. (2020): Finite and Infinite Integral Formulas Involving the Family of Incomplete *H*-Functions, AAM: Intern. J., 6, p. 15-28
3. M. K. Bansal and J. Choi (2019). A note on pathway fractional integral formulas associated with the incomplete *H*-Functions, Int. *J. Appl. Comput. Math,* **5** Art. 133. [CrossRef]
4. M. K. Bansal, N.Jolly and R.Jain et al. (2019). An integral operator involving generalized Mittag-Leffer function and associated fractional calculus results, *J. Anal.*, **27**, 727–740. [CrossRef]
5. M. K. Bansal and D.Kumar and R Jain (2019), Interrelationships between Marichev-Saigo-Maeda fractional integral operators, the Laplace transform and the $\bar{H}$ -Function, *Int. J. Appl. Comput. Math*,**5,** Art. 103. [CrossRef]
6. M. K. Bansal and D.Kumar and I.Khan et al. (2019). Certain unified integrals associated withproduct of M-series and incomplete *H*-functions, *Mathematics*, **7,** 1191. [CrossRef]
7. M.A.Chaudhry and A.Qadir (2002). Incomplete exponential and hypergeometric functions with applications to non-central χ^2-Distribution, *Comm. Statist. Theory Methods*,**34**, 525–535. [CrossRef]
8. C.Fox (1961). The G and H-Functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.*,**98**, 395-429. [CrossRef]
9. A.M.Mathai, R.K.Saxena and H.J.Houbold (2010). *The H-Function; Theoryand Applications,* Springer, New York.
10. V.P. Saxena (2008). *TheI-function*, Anamaya Publisher, New Delhi. [CrossRef]
11. R. Srivastava and N.E. Cho (2012). Generating functions for a certain class of incomplete hypergeometric polynomials, *Appl. Math. Comput.***219**, 3219–3225. [CrossRef]
12. R. Srivastava (2013). Some properties of a family of incomplete hypergeometric functions, *Russian J. Math. Phys*., **20**, 121–128. [CrossRef]
13. H.M. Srivastava,M.A. Chaudhary and R.P. Agarwal (2012). The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec.Funct.,**23**, 659–683. [CrossRef]
14. H. M.Srivastava, R. K.Saxena and R. K.Parmar (2018). Some families of theincomplete *H*-Functions and the incomplete $\bar{H}$-Functions and associated integral transforms and operators of fractional calculus with applications, *Russ. J. Math. Phys.*,**25,** 116–138.
15. H. M.Srivastava, K. C.Gupta and S. P. Goyal (1982). The *H*-Functions ofone and two variables with applications, *South Asian Publishers, New Delhi and Madras.*

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